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YET ANOTHER AIRFOIL

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The Joukowski airfoil is obtained by applying the transformation $z + \frac{1}{z}$ to a circle. It is a quartic curve. Here we describe another airfoil; it also is a quartic curve given by the equation AF:

$$m^2(x^2+y^2)^2 - 2(x^2+y^2)(Ax+By) + (Cx+Dy)^2 = 0$$

$$A = m(mu-v), B = (2+m^2)v - mu, C = mu - v, D = -(u+mv).$$

Here is how our curve is obtained. Given two lines L, M meeting at O and a point P. For every line N through P make the circle through the three points O, and the two intersections Q, R of N with L, M. Our airfoil is the envelope of this family of circles.

Let us choose coordinates so that O is at the origin; $P = (u, v)$, L is the x-axis and M has equation $y = mx$.

Suppose that the line N meets the x-axis at $(t, 0)$ then the equation for N is $y = \frac{v}{u-t}(x-t)$ and it meets M when $mx = \frac{v}{u-t}(x-t)$. Thus the equation of the circle in the family for this line is $x^2 + y^2 - tx - sy$, where $s = \frac{t(mv+u-t)}{m(t-u)+v}$.

The envelope of a family of curves $f(x, y, t)$ depending on a parameter t has an equation obtained by elimination of t from the system $F = f(x, y, t) = 0, G = \frac{\partial f}{\partial t} f(x, y, t) = 0$. We can easily eliminate t from these by computing the discriminant of F with respect to t giving the equation AF above.

This curve has a singularity at the origin and the singular tangents are the factors of the quadratic terms. Since the quadratic term is squared it is a cusp. The line $y = mx$ is also tangent and the curve is tangent to the x-axis.

We can rewrite this equation in terms of complex coordinates $z = x + iy$ in order to obtain

$$m^2(z\bar{z})^2 + z\bar{z}(\bar{b}z + b\bar{z}) + 2(\bar{f}z^2 + ez\bar{z} + f\bar{z}^2) = 0.$$

Solving we obtain $b = mv - m^2u + I(2v + m^2v - mu)$, $2e = (1 + m^2)(u^2 + v^2)$, $2f = \frac{1}{2}(u^2 - v^2)(m^2 - 1) - 2muv + I(m(v^2 - u^2) + (1 - m^2)vu)$.

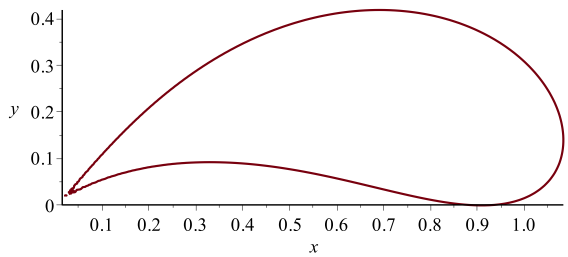

FIGURE 1. $m = 1.1, u = 1, v = .1$

Using the Maple plot package we obtain the two Figures.

```
implicitplot(subs(m=1.1,u=1,v=.1,AF),x=0..3/2,y=0..1,numpoints=20000)
implicitplot(subs(m=1.1,u=2,v=.3,AF),x=0..3,y=0..1,numpoints=20000)
```

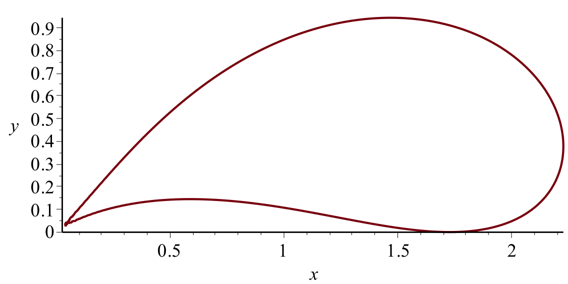

FIGURE 2. $m = 1.1, u = 2, v = .3$

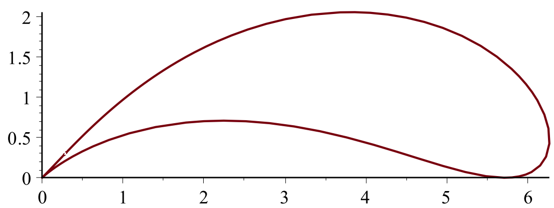

FIGURE 3. $m = 1.05, u = 6, v = .3$

Parameterization Let

$$x = \frac{vt^2(mt^2 - 2(mu - v)t + (mu - v)(u + mv))}{m^2t^4 - 4m(mu - v)t^3 + 2(3mu - 2v)(mu - v)t^2 - 4u(mu - v)^2t + (u^2 + v^2)(mu - v)}$$

$$y = \frac{vt^2(mt - mu + v)^2}{m^2t^4 - 4m(mu - v)t^3 + 2(3mu - 2v)(mu - v)t^2 - 4u(mu - v)^2t + (u^2 + v^2)(mu - v)}$$

then these satisfy the equation $AF = 0$.

With Maple

```
p:=subs(m=1.05, u=6, v=.3,x);
q:=subs(m=1.05, u=6, v=.3,y);
plot([ p,q, t=-300..300], numpoints=3000)
```

we obtain Figure 3.

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